

Chapter 9

Inverse Trigonometric Functions

1. If x, y, z are in A.P. and $\tan^{-1}x, \tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then [JEE (Main)-2013]
- (1) $x = y = z$ (2) $2x = 3y = 6z$
(3) $6x = 3y = 2z$ (4) $6x = 4y = 3z$
2. If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to [JEE (Main)-2013]
- (1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{2}$
(3) 1 (4) $\sqrt{2}$
3. Let $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ where $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is [JEE (Main)-2015]
- (1) $\frac{3x-x^3}{1-3x^2}$ (2) $\frac{3x+x^3}{1-3x^2}$
(3) $\frac{3x-x^3}{1+3x^2}$ (4) $\frac{3x+x^3}{1+3x^2}$
4. Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x \in \left(0, \frac{\pi}{2}\right)$. A normal to $y = f(x)$ at $x = \frac{\pi}{6}$ also passes through the point [JEE (Main)-2016]
- (1) $\left(0, \frac{2\pi}{3}\right)$ (2) $\left(\frac{\pi}{6}, 0\right)$
(3) $\left(\frac{\pi}{4}, 0\right)$ (4) $(0, 0)$
5. If $\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2} \left(x > \frac{3}{4}\right)$, then x is equal to [JEE (Main)-2019]
- (1) $\frac{\sqrt{146}}{12}$ (2) $\frac{\sqrt{145}}{12}$
(3) $\frac{\sqrt{145}}{10}$ (4) $\frac{\sqrt{145}}{11}$
6. If $x = \sin^{-1}(\sin 10)$ and $y = \cos^{-1}(\cos 10)$, then $y - x$ is equal to [JEE (Main)-2019]
- (1) 7π (2) 10
(3) 0 (4) π
7. The value of $\cot\left(\sum_{n=1}^{19} \cot^{-1}\left(1 + \sum_{p=1}^n 2p\right)\right)$ is [JEE (Main)-2019]
- (1) $\frac{19}{21}$ (2) $\frac{23}{22}$
(3) $\frac{22}{23}$ (4) $\frac{21}{19}$
8. All x satisfying the inequality $(\cot^{-1}x)^2 - 7(\cot^{-1}x) + 10 > 0$, lie in the interval [JEE (Main)-2019]
- (1) $(\cot 2, \infty)$
(2) $(\cot 5, \cot 4)$
(3) $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$
(4) $(-\infty, \cot 5) \cup (\cot 2, \infty)$
9. If $\alpha = \cos^{-1}\left(\frac{3}{5}\right), \beta = \tan^{-1}\left(\frac{1}{3}\right)$, where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to [JEE (Main)-2019]
- (1) $\tan^{-1}\left(\frac{9}{14}\right)$ (2) $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$
(3) $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (4) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

10. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, where $-1 \leq x \leq 1$, $-2 \leq y \leq 2$, $x \leq \frac{y}{2}$, then for all x, y , $4x^2 - 4xy \cos \alpha + y^2$ is equal to : **[JEE (Main)-2019]**

- (1) $2 \sin^2 \alpha$ (2) $4 \sin^2 \alpha - 2x^2 y^2$
(3) $4 \cos^2 \alpha + 2x^2 y^2$ (4) $4 \sin^2 \alpha$

11. The value of $\sin^{-1}\left(\frac{12}{13}\right) - \sin^{-1}\left(\frac{3}{5}\right)$ is equal to

[JEE (Main)-2019]

- (1) $\frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$ (2) $\pi - \sin^{-1}\left(\frac{63}{65}\right)$
(3) $\pi - \cos^{-1}\left(\frac{33}{65}\right)$ (4) $\frac{\pi}{2} - \cos^{-1}\left(\frac{9}{65}\right)$

12. If $f(x) = \tan^{-1}(\sec x + \tan x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and $f(0) = 0$, then $f(1)$ is equal to **[JEE (Main)-2020]**

- (1) $\frac{\pi+1}{4}$ (2) $\frac{1}{4}$
(3) $\frac{\pi+2}{4}$ (4) $\frac{\pi-1}{4}$

13. The domain of the function

$$f(x) = \sin^{-1}\left(\frac{|x|+5}{x^2+1}\right) \text{ is}$$

$(-\infty, -a] \cup [a, \infty)$. Then a is equal to

[JEE (Main)-2020]

- (1) $\frac{1+\sqrt{17}}{2}$ (2) $\frac{\sqrt{17}}{2} + 1$
(3) $\frac{\sqrt{17}-1}{2}$ (4) $\frac{\sqrt{17}}{2}$

14. $2\pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}\right)$ is equal to

[JEE (Main)-2020]

- (1) $\frac{\pi}{2}$ (2) $\frac{7\pi}{4}$
(3) $\frac{3\pi}{2}$ (4) $\frac{5\pi}{4}$

15. If S is the sum of the first 10 terms of the series

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \tan^{-1}\left(\frac{1}{21}\right) + \dots,$$

then $\tan(S)$ is equal to

[JEE (Main)-2020]

- (1) $-\frac{6}{5}$ (2) $\frac{5}{11}$
(3) $\frac{10}{11}$ (4) $\frac{5}{6}$

16. The derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to

$$\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right) \text{ at } x = \frac{1}{2} \text{ is}$$

[JEE (Main)-2020]

- (1) $\frac{2\sqrt{3}}{3}$ (2) $\frac{2\sqrt{3}}{5}$
(3) $\frac{\sqrt{3}}{10}$ (4) $\frac{\sqrt{3}}{12}$

17. If $y = \sum_{k=1}^6 k \cos^{-1}\left\{\frac{3}{5} \cos kx - \frac{4}{5} \sin kx\right\}$, then

$$\frac{dy}{dx} \text{ at } x = 0 \text{ is } \underline{\hspace{2cm}}.$$

[JEE (Main)-2020]

18. $\lim_{n \rightarrow \infty} \tan^{-1}\left\{\sum_{r=1}^n \tan^{-1}\left(\frac{1}{1+r+r^2}\right)\right\}$ is equal to

[JEE (Main)-2021]

19. A possible value of $\tan\left(\frac{1}{4} \sin^{-1}\frac{\sqrt{63}}{8}\right)$ is :

[JEE (Main)-2021]

- (1) $2\sqrt{2}-1$ (2) $\sqrt{7}-1$
(3) $\frac{1}{\sqrt{7}}$ (4) $\frac{1}{2\sqrt{2}}$

20. $\operatorname{cosec}\left[2 \cot^{-1}(5) + \cos^{-1}\left(\frac{4}{5}\right)\right]$ is equal to :

[JEE (Main)-2021]

- (1) $\frac{65}{56}$ (2) $\frac{65}{33}$
(3) $\frac{75}{56}$ (4) $\frac{56}{33}$

21. If $\frac{\sin^{-1} x}{a} = \frac{\cos^{-1} x}{b} = \frac{\tan^{-1} y}{c}$; $0 < x < 1$, then the

value of $\cos\left(\frac{\pi c}{a+b}\right)$ is : **[JEE (Main)-2021]**

(1) $1 - y^2$ (2) $\frac{1-y^2}{y\sqrt{y}}$

(3) $\frac{1-y^2}{1+y^2}$ (4) $\frac{1-y^2}{2y}$

22. If $0 < a, b < 1$, and $\tan^{-1} a + \tan^{-1} b = \frac{\pi}{4}$, then the value of

$(a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) - \left(\frac{a^4+b^4}{4}\right) + \dots$ is :

[JEE (Main)-2021]

(1) $e^2 - 1$ (2) $\log_e \left(\frac{e}{2}\right)$

(3) e (4) $\log_e 2$

23. Given that the inverse trigonometric function take principal values only. Then, the number of real values of x which satisfy

$\sin^{-1}\left(\frac{3x}{5}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1} x$ is equal to :

[JEE (Main)-2021]

(1) 3 (2) 1

(3) 0 (4) 2

24. If $\cot^{-1}(\alpha) = \cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cot^{-1}32 + \dots$ upto 100 terms, then α is :

[JEE (Main)-2021]

(1) 1.01 (2) 1.02

(3) 1.03 (4) 1.00

25. The sum of possible values of x for $\tan^{-1}(x+1) + \cot^{-1}\left(\frac{1}{x-1}\right) = \tan^{-1}\left(\frac{8}{31}\right)$ is

[JEE (Main)-2021]

(1) $-\frac{30}{4}$ (2) $-\frac{31}{4}$

(3) $-\frac{32}{4}$ (4) $-\frac{33}{4}$

26. The number of solutions of the equation

$\sin^{-1}\left[x^2 + \frac{1}{3}\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] = x^2$, for $x \in [-1, 1]$

and $[x]$ denotes the greatest integer less than or equal to x , is : **[JEE (Main)-2021]**

(1) Infinite (2) 2

(3) 4 (4) 0

27. The real valued function $f(x) = \frac{\operatorname{cosec}^{-1} x}{\sqrt{x - [x]}}$, where $[x]$

denotes the greatest integer less than or equal to x , is defined for all x belonging to :

[JEE (Main)-2021]

(1) all non-integers except the interval $[-1, 1]$

(2) all integers except 0, -1, 1

(3) all reals except integers

(4) all reals except the interval $[-1, 1]$

28. The number of real roots of the equation

$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{4}$ is :

[JEE (Main)-2021]

(1) 2 (2) 1

(3) 4 (4) 0

29. The value of $\tan\left(2\tan^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right)\right)$ is equal to

[JEE (Main)-2021]

(1) $\frac{220}{21}$ (2) $\frac{151}{63}$

(3) $\frac{-181}{69}$ (4) $\frac{-291}{76}$

30. The domain of the function $\operatorname{cosec}^{-1}\left(\frac{1+x}{x}\right)$ is :

[JEE (Main)-2021]

(1) $\left[-\frac{1}{2}, 0\right) \cup [1, \infty)$ (2) $\left[-\frac{1}{2}, \infty\right) - \{0\}$

(3) $\left(-1, -\frac{1}{2}\right] \cup (0, \infty)$ (4) $\left(-\frac{1}{2}, \infty\right) - \{0\}$

31. If $(\sin^{-1} x)^2 - (\cos^{-1} x)^2 = a$; $0 < x < 1$, $a \neq 0$, then the value of $2x^2 - 1$ is **[JEE (Main)-2021]**

(1) $\cos\left(\frac{2a}{\pi}\right)$ (2) $\sin\left(\frac{4a}{\pi}\right)$

(3) $\cos\left(\frac{4a}{\pi}\right)$ (4) $\sin\left(\frac{2a}{\pi}\right)$

32. If $y(x) = \cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right)$, $x \in \left(\frac{\pi}{2}, \pi \right)$,

then $\frac{dy}{dx}$ at $x = \frac{5\pi}{6}$ is [JEE (Main)-2021]

(1) $-\frac{1}{2}$ (2) 0

(3) -1 (4) $\frac{1}{2}$

33. Let M and m respectively be the maximum and minimum values of the function $f(x) = \tan^{-1}(\sin x + \cos x)$ in $\left[0, \frac{\pi}{2} \right]$. Then the value of $\tan(M - m)$ is equal to [JEE (Main)-2021]

(1) $2 - \sqrt{3}$ (2) $2 + \sqrt{3}$

(3) $3 + 2\sqrt{2}$ (4) $3 - 2\sqrt{2}$

34. The domain of the function

$f(x) = \sin^{-1} \left(\frac{3x^2 + x - 1}{(x-1)^2} \right) + \cos^{-1} \left(\frac{x-1}{x+1} \right)$ is [JEE (Main)-2021]

(1) $\left[0, \frac{1}{4} \right]$ (2) $\left[0, \frac{1}{2} \right]$

(3) $\left[\frac{1}{4}, \frac{1}{2} \right] \cup \{0\}$ (4) $[-2, 0] \cup \left[\frac{1}{4}, \frac{1}{2} \right]$

35. $\cos^{-1}(\cos(-5)) + \sin^{-1}(\sin(6)) - \tan^{-1}(\tan(12))$ is equal to (The inverse trigonometric functions take the principal values) [JEE (Main)-2021]

(1) $3\pi + 1$ (2) $3\pi - 11$
(3) $4\pi - 11$ (4) $4\pi - 9$

36. Let $S_k = \sum_{r=1}^k \tan^{-1} \left(\frac{6^r}{2^{2r+1} + 3^{2r+1}} \right)$. Then $\lim_{k \rightarrow \infty} S_k$ is equal to : [JEE (Main)-2021]

(1) $\tan^{-1}(3)$ (2) $\tan^{-1} \left(\frac{3}{2} \right)$

(3) $\cot^{-1} \left(\frac{3}{2} \right)$ (4) $\frac{\pi}{2}$

37. If $\sum_{r=1}^{50} \tan^{-1} \frac{1}{2r^2} = p$, then the value of $\tan p$ is [JEE (Main)-2021]

(1) 100 (2) $\frac{50}{51}$

(3) $\frac{101}{102}$ (4) $\frac{51}{50}$

38. Considering only the principal values of inverse functions, the set [JEE (Main)-2021]

$A = \left\{ x \geq 0 : \tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4} \right\}$

(1) Is a singleton

(2) Contains two elements

(3) Contains more than two elements

(4) Is an empty set

39. The set of all values of k for which

$(\tan^{-1} x)^3 + (\cot^{-1} x)^3 = k\pi^3$, $x \in \mathbb{R}$, is the interval [JEE (Main)-2022]

(1) $\left[\frac{1}{32}, \frac{7}{8} \right]$ (2) $\left(\frac{1}{24}, \frac{13}{16} \right)$

(3) $\left[\frac{1}{48}, \frac{13}{16} \right]$ (4) $\left[\frac{1}{32}, \frac{9}{8} \right]$

40. The domain of the function

$f(x) = \frac{\cos^{-1} \left(\frac{x^2 - 5x + 6}{x^2 - 9} \right)}{\log_e(x^2 - 3x + 2)}$ is [JEE (Main)-2022]

(1) $(-\infty, 1) \cup (2, \infty)$

(2) $(2, \infty)$

(3) $\left[-\frac{1}{2}, 1 \right) \cup (2, \infty)$

(4) $\left[-\frac{1}{2}, 1 \right) \cup (2, \infty) - \left\{ \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2} \right\}$

41. Let $x * y = x^2 + y^3$ and $(x * 1) * 1 = x * (1 * 1)$.

Then a value of $2 \sin^{-1} \left(\frac{x^4 + x^2 - 2}{x^4 + x^2 + 2} \right)$ is [JEE (Main)-2022]

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{3}$

(3) $\frac{\pi}{2}$ (4) $\frac{\pi}{6}$

42. The value of $\tan^{-1} \left(\frac{\cos\left(\frac{15\pi}{4}\right) - 1}{\sin\left(\frac{\pi}{4}\right)} \right)$ is equal to

[JEE (Main)-2022]

(1) $-\frac{\pi}{4}$ (2) $-\frac{\pi}{8}$

(3) $-\frac{5\pi}{12}$ (4) $-\frac{4\pi}{9}$

43. Let $f(x) = 2\cos^{-1}x + 4\cot^{-1}x - 3x^2 - 2x + 10$, $x \in [-1, 1]$, If $[a, b]$ is the range of the function f , then $4a - b$ is equal to

[JEE (Main)-2022]

(1) 11 (2) $11 - \pi$
(3) $11 + \pi$ (4) $15 - \pi$

44. If the inverse trigonometric functions take principal values, then

$\cos^{-1} \left(\frac{3}{10} \cos \left(\tan^{-1} \left(\frac{4}{3} \right) \right) + \frac{2}{5} \sin \left(\tan^{-1} \left(\frac{4}{3} \right) \right) \right)$

is equal to

[JEE (Main)-2022]

(1) 0 (2) $\frac{\pi}{4}$

(3) $\frac{\pi}{3}$ (4) $\frac{\pi}{6}$

45. $\sin^{-1} \left(\sin \frac{2\pi}{3} \right) + \cos^{-1} \left(\cos \frac{7\pi}{6} \right) + \tan^{-1} \left(\tan \frac{3\pi}{4} \right)$ is equal to

[JEE (Main)-2022]

(1) $\frac{11\pi}{12}$ (2) $\frac{17\pi}{12}$

(3) $\frac{31\pi}{12}$ (4) $-\frac{3\pi}{4}$

46. The value of $\cot \left(\sum_{n=1}^{50} \tan^{-1} \left(\frac{1}{1+n+n^2} \right) \right)$ is

[JEE (Main)-2022]

(1) $\frac{26}{25}$ (2) $\frac{25}{26}$

(3) $\frac{50}{51}$ (4) $\frac{52}{51}$

47. The value of $\lim_{n \rightarrow \infty} 6 \tan \left\{ \sum_{r=1}^n \tan^{-1} \left(\frac{1}{r^2 + 3r + 3} \right) \right\}$

is equal to

[JEE (Main)-2022]

(1) 1 (2) 2
(3) 3 (4) 6

48. The domain of the function

$\cos^{-1} \left(\frac{2 \sin^{-1} \left(\frac{1}{4x^2 - 1} \right)}{\pi} \right)$ is

[JEE (Main)-2022]

(1) $\mathbb{R} - \left\{ -\frac{1}{2}, \frac{1}{2} \right\}$

(2) $(-\infty, -1] \cup [1, \infty) \cup \{0\}$

(3) $\left(-\infty, -\frac{1}{2} \right) \cup \left(\frac{1}{2}, \infty \right) \cup \{0\}$

(4) $\left(-\infty, \frac{1}{\sqrt{2}} \right] \cup \left[\frac{1}{\sqrt{2}}, \infty \right) \cup \{0\}$

49. $50 \tan \left(3 \tan^{-1} \left(\frac{1}{2} \right) + 2 \cos^{-1} \left(\frac{1}{\sqrt{5}} \right) \right)$

$+ 4\sqrt{2} \tan \left(\frac{1}{2} \tan^{-1} (2\sqrt{2}) \right)$ is equal to _____.

[JEE (Main)-2022]

50. $\tan \left(2 \tan^{-1} \frac{1}{5} + \sec^{-1} \frac{\sqrt{5}}{2} + 2 \tan^{-1} \frac{1}{8} \right)$ is equal to

[JEE (Main)-2022]

(1) 1 (2) 2

(3) $\frac{1}{4}$ (4) $\frac{5}{4}$

51. For $k \in \mathbb{R}$, let the solution of the equation

$\cos \left(\sin^{-1} \left(x \cot \left(\tan^{-1} \left(\cos \left(\sin^{-1} \right) \right) \right) \right) \right) = k, 0 < |x| < \frac{1}{\sqrt{2}}$

Inverse trigonometric functions take only principal values. If the solutions of the equation $x^2 - bx - 5 = 0$

are $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ and $\frac{\alpha}{\beta}$, then $\frac{b}{k^2}$ is equal to _____.

[JEE (Main)-2022]

52. Considering only the principal values of the inverse trigonometric functions, the domain of the function

$$f(x) = \cos^{-1}\left(\frac{x^2 - 4x + 2}{x^2 + 3}\right) \text{ is } \quad \text{[JEE (Main)-2022]}$$

- (1) $\left(-\infty, \frac{1}{4}\right]$ (2) $\left[-\frac{1}{4}, \infty\right)$
 (3) $\left(\frac{-1}{3}, \infty\right)$ (4) $\left(-\infty, \frac{1}{3}\right]$

53. Considering the principal values of the inverse trigonometric functions, the sum of all the solutions of the equation $\cos^{-1}(x) - 2\sin^{-1}(x) = \cos^{-1}(2x)$ is equal to
 [JEE (Main)-2022]

- (1) 0 (2) 1
 (3) $\frac{1}{2}$ (4) $-\frac{1}{2}$

54. The sum of the absolute maximum and absolute minimum values of the function

$$f(x) = \tan^{-1}(\sin x - \cos x) \text{ in the interval } [0, \pi] \text{ is}$$

- (1) 0 (2) $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi}{4}$
 (3) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) - \frac{\pi}{4}$ (4) $-\frac{\pi}{12}$

[JEE (Main)-2022]

55. The domain of the function

$$f(x) = \sin^{-1}\left(\frac{x^2 - 3x + 2}{x^2 + 2x + 7}\right) \text{ is}$$

- (1) $[1, \infty)$ (2) $[-1, 2]$
 (3) $[-1, \infty)$ (4) $(-\infty, 2]$

[JEE (Main)-2022]

56. If $0 < x < \frac{1}{\sqrt{2}}$ and $\frac{\sin^{-1} x}{\alpha} = \frac{\cos^{-1} x}{\beta}$, then a value

$$\text{of } \sin\left(\frac{2\pi\alpha}{\alpha + \beta}\right) \text{ is}$$

- (1) $4\sqrt{(1-x^2)}(1-2x^2)$ (2) $4x\sqrt{(1-x^2)}(1-2x^2)$
 (3) $2x\sqrt{(1-x^2)}(1-4x^2)$ (4) $4\sqrt{(1-x^2)}(1-4x^2)$

[JEE (Main)-2022]

57. The domain of the function

$$f(x) = \sin^{-1}[2x^2 - 3] + \log_2\left(\log_{\frac{1}{2}}\left(x^2 - 5x + 5\right)\right),$$

where $[t]$ is the greatest integer function, is

[JEE (Main)-2022]

- (1) $\left(-\sqrt{\frac{5}{2}}, \frac{5-\sqrt{5}}{2}\right)$ (2) $\left(\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right)$
 (3) $\left(1, \frac{5-\sqrt{5}}{2}\right)$ (4) $\left(1, \frac{5+\sqrt{5}}{2}\right)$

58. Let $x = \sin(2\tan^{-1} \alpha)$ and $y = \sin\left(\frac{1}{2}\tan^{-1} \frac{4}{3}\right)$. If

$$S = \{\alpha \in \mathbb{R} : y^2 = 1 - x\}, \text{ then } \sum_{\alpha \in S} 16\alpha^3 \text{ is equal to}$$

_____.

[JEE (Main)-2022]



Chapter 9

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1. Answer (1)

$$\therefore 2y = x + z \text{ and } 2\tan^{-1}y = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$$

$$\Rightarrow \frac{2y}{1-y^2} = \frac{x+z}{1-xy}$$

$$\Rightarrow \boxed{y^2 = xz}$$

$\Rightarrow x, y, z$ are in GP

$$\therefore x = y = z$$

2. Answer (1)

$$y = \sec \sec^{-1}(\sqrt{1+x^2}) = \sqrt{1+x^2}$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$$

$$\left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{\sqrt{2}}$$

3. Answer (1)

$$\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$y = \frac{3x-x^3}{1-3x^2}$$

4. Answer (1)

$$f(x) = \tan^{-1} \frac{1+\sin x}{1-\sin x}$$

$$= \tan^{-1} \frac{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)^2}{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)^2}$$

$$= \tan^{-1} \left(\tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right)$$

$$= \frac{\pi}{4} + \frac{x}{2}$$

$$\Rightarrow f'(x) = \frac{1}{2} \text{ and at } x = \frac{\pi}{6}, f(x) = \frac{\pi}{3}$$

So, equation of normal is

$$y - \frac{\pi}{3} = -2 \left(x - \frac{\pi}{6} \right) \Rightarrow y + 2x = \frac{2\pi}{3}$$

5. Answer (2)

$$\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2} \quad \left(x > \frac{3}{4}\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{3}{4x}\right)$$

$$\therefore \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) = \sin^{-1}\left(\frac{3}{4x}\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) = \cos^{-1}\left(\frac{\sqrt{16x^2-9}}{4x}\right)$$

$$\therefore \sin^{-1}\left(\frac{3}{4x}\right) = \cos^{-1}\left(\frac{\sqrt{16x^2-9}}{4x}\right)$$

$$\Rightarrow \frac{2}{3x} = \frac{\sqrt{16x^2-9}}{4x}$$

$$\Rightarrow x^2 = \frac{64+81}{9 \times 16}$$

$$\boxed{x = \frac{\sqrt{145}}{12}} \quad \left(\because x > \frac{3}{4}\right)$$

6. Answer (4)

$$x = \sin^{-1}(\sin 10)$$

$$x = 3\pi - 10 \quad \begin{cases} 3\pi - \frac{\pi}{2} < 10 < 3\pi + \frac{\pi}{2} \\ \Rightarrow 3\pi - x \end{cases}$$

$$\text{and } y = \cos^{-1}(\cos 10) \quad \begin{cases} 3\pi < 10 < 4\pi \\ \Rightarrow 4\pi - x \end{cases}$$

$$y = 4\pi - 10$$

$$\therefore y - x = (4\pi - 10) - (3\pi - 10) = \pi$$

7. Answer (4)

$$\cot \left(\sum_{n=1}^{19} \cot^{-1} \left(1 + \sum_{p=1}^n 2p \right) \right)$$

$$= \cot \left(\sum_{n=1}^{19} \cot^{-1} (1 + n(n+1)) \right)$$

$$= \cot \left(\sum_{n=1}^{19} \tan^{-1} \left(\frac{(n+1) - n}{1 + (n+1)n} \right) \right)$$

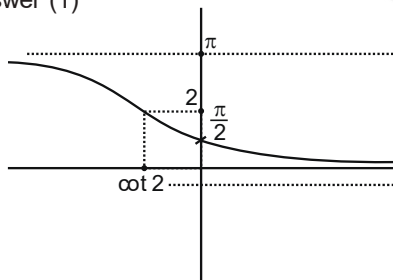
$$= \cot \left(\sum_{n=1}^{19} (\tan^{-1}(n+1) - \tan^{-1} n) \right)$$

$$= \cot (\tan^{-1} 20 - \tan^{-1} 1)$$

$$= \cot \left(\tan^{-1} \left(\frac{20-1}{1+20 \times 1} \right) \right)$$

$$= \cot \left(\tan^{-1} \left(\frac{19}{21} \right) \right) = \cot \cot^{-1} \left(\frac{21}{19} \right) = \frac{21}{19}$$

8. Answer (1)



$$(\cot^{-1} x - 5)(\cot^{-1} 2) > 0$$

$$\cot^{-1} x \in (-\infty, 2) \cup (5, \infty) \quad \dots(i)$$

But $\cot^{-1} x$ lies in $(0, \pi)$

From equation (i)

So, $\cot^{-1} x \in (0, 2)$

By graph,

$$x \in (\cot 2, \infty)$$

9. Answer (3)

$$\therefore \cos \alpha = \frac{3}{5}$$

$$\Rightarrow \tan \alpha = \frac{4}{3}$$

$$\Rightarrow \text{and } \tan \beta = \frac{1}{3}$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \cdot \tan \beta}$$

$$= \frac{\frac{4}{3} - \frac{1}{3}}{1 + \frac{4}{3} \cdot \frac{1}{3}} = \frac{1}{\frac{13}{9}}$$

$$= \frac{9}{13}$$

$$\alpha - \beta = \tan^{-1} \left(\frac{9}{13} \right) = \sin^{-1} \left(\frac{9}{5\sqrt{10}} \right) = \cos^{-1} \left(\frac{13}{5\sqrt{10}} \right)$$

10. Answer (4)

$$\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$$

$$\Rightarrow \cos^{-1} \left(\frac{xy}{2} + \sqrt{1-x^2} \sqrt{1-\frac{y^2}{4}} \right) = \alpha$$

$$\Rightarrow \frac{xy}{2} + \frac{\sqrt{1-x^2} \sqrt{4-y^2}}{2} = \cos \alpha$$

$$\Rightarrow xy + \sqrt{1-x^2} \sqrt{4-y^2} = 2 \cos \alpha$$

$$(xy - 2 \cos \alpha)^2 = (1-x^2)(4-y^2)$$

$$x^2 y^2 + 4 \cos^2 \alpha - 4xy \cos \alpha = 4 - y^2 - 4x^2 + x^2 y^2$$

$$4x^2 - 4xy \cos \alpha + y^2 = 4 \sin^2 \alpha$$

11. Answer (1)

$$-\sin^{-1} \left(\frac{3}{5} \right) + \sin^{-1} \left(\frac{12}{13} \right)$$

$$= -\sin^{-1} \left(\frac{3}{5} \times \frac{5}{13} - \frac{12}{13} \times \frac{4}{5} \right)$$

$$(\because xy \geq 0 \text{ and } x^2 + y^2 \leq 1)$$

$$= -\sin^{-1} \left(\frac{-33}{65} \right) = \sin^{-1} \left(\frac{33}{65} \right)$$

$$= \cos^{-1} \left(\frac{56}{65} \right) = \frac{\pi}{2} - \sin^{-1} \left(\frac{56}{65} \right)$$

12. Answer (1)

$$f'(x) = \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$$

$$\therefore \frac{\pi}{4} + \frac{x}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ in the neighbourhood of } x = 1$$

$$\Rightarrow f'(x) = \frac{\pi}{4} + \frac{x}{2}$$

$$\Rightarrow f(x) = \frac{\pi}{4}x + \frac{x^2}{4} + c$$

$$\therefore f(0) = 0 \Rightarrow c = 0$$

$$\text{So } f(1) = \frac{\pi}{4} + \frac{1}{4} = \frac{\pi+1}{4}$$

13. Answer (1)

$$\therefore f(x) = \sin^{-1}\left(\frac{|x|+5}{x^2+1}\right)$$

$$\therefore -1 \leq \frac{|x|+5}{x^2+1} \leq 1$$

$$\therefore x^2 - |x| - 4 \geq 0$$

$$\left(|x| - \frac{1-\sqrt{17}}{2}\right)\left(|x| - \frac{1+\sqrt{17}}{2}\right) \geq 0$$

$$\therefore |x| \geq \frac{1+\sqrt{17}}{2}$$

$$\therefore x \in \left(-\infty, -\frac{1+\sqrt{17}}{2}\right] \cup \left[\frac{1+\sqrt{17}}{2}, \infty\right)$$

14. Answer (3)

$$2\pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}\right)$$

$$= 2\pi - \left(\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{16}{63}\right)$$

$$= 2\pi - \left\{ \tan^{-1}\left(\frac{\frac{4}{3} + \frac{5}{12}}{1 - \frac{4}{3} \cdot \frac{5}{12}}\right) + \tan^{-1}\frac{16}{63} \right\}$$

$$= 2\pi - \left(\tan^{-1}\frac{63}{16} + \tan^{-1}\frac{16}{63}\right)$$

$$= 2\pi - \left(\tan^{-1}\frac{63}{16} + \cot^{-1}\frac{63}{16}\right)$$

$$= 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$$

15. Answer (4)

$$S = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \dots$$

$$= \tan^{-1}\left(\frac{1}{1+1 \cdot 2}\right) + \tan^{-1}\left(\frac{1}{1+2 \cdot 3}\right) + \tan^{-1}\left(\frac{1}{1+3 \cdot 4}\right) + \dots$$

$$= \tan^{-1}\left(\frac{2-1}{1+2 \cdot 1}\right) + \tan^{-1}\left(\frac{3-2}{1+3 \cdot 2}\right) +$$

$$\tan^{-1}\left(\frac{4-3}{1+3 \cdot 4}\right) + \dots + \tan^{-1}\left(\frac{11-10}{1+11 \cdot 10}\right)$$

$$= (\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) +$$

$$(\tan^{-1}4 - \tan^{-1}3) + \dots + (\tan^{-1}11 - \tan^{-1}10)$$

$$= \tan^{-1}11 - \tan^{-1}1$$

$$= \tan^{-1}\left(\frac{11-1}{1+11 \cdot 1}\right)$$

$$= \tan^{-1}\left(\frac{5}{6}\right)$$

$$\tan(S) = \frac{5}{6}$$

16. Answer (3)

$$\frac{d\left(\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right)}{d\left(\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)\right)} = \frac{d\left(\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right)}{d\left(\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)\right)} \cdot \frac{dx}{dx}$$

$$\text{Simplifying } \left(\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right) \quad \text{Put } x = \tan\theta$$

$$\Rightarrow \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-(1-2\sin^2\theta/2)}{2\sin\theta/2 \cos\theta/2}\right) = \frac{\theta}{2}$$

$$\therefore \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{\tan^{-1}x}{2}$$

$$\& \text{ similarly } \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right) \quad \text{Put } x = \sin\theta$$

$$\Rightarrow \tan^{-1}\left(\frac{\sin 2\theta}{\cos 2\theta}\right) = 2\theta = 2\sin^{-1}x$$

Hence required derivative

$$\frac{\frac{1}{2(1+x^2)}}{\frac{2}{\sqrt{1-x^2}}} = \frac{\sqrt{1-x^2}}{4(1+x^2)} \bigg|_{x=\frac{1}{2}} = \frac{\sqrt{3}}{10}$$

17. Answer (91)

$$y = \sum_{k=1}^6 k \cos^{-1} \left\{ \frac{3}{5} \cos kx - \frac{4}{5} \sin kx \right\}$$

$$\text{Let } \cos \alpha = \frac{3}{5} \text{ and } \sin \alpha = \frac{4}{5}$$

$$\therefore y = \sum_{k=1}^6 k \cos^{-1} \{ \cos \alpha \cos kx - \sin \alpha \sin kx \}$$

$$= \sum_{k=1}^6 k \cos^{-1} (\cos(kx + \alpha))$$

$$= \sum_{k=1}^6 k (kx + \alpha) = \sum_{k=1}^6 (k^2 x + \alpha k)$$

$$\frac{dy}{dx} = \sum_{k=1}^6 k^2 = \frac{6(7)(13)}{6} = 91$$

18. Answer (1)

$$\tan^{-1} \left(\frac{1}{1+r+r^2} \right) = \tan^{-1} \left(\frac{(r+1)-r}{1+(r+1)r} \right)$$

$$= \tan^{-1}(r+1) - \tan^{-1}r$$

$$\text{So } \sum_{r=1}^n \tan^{-1} \left(\frac{1}{1+r+r^2} \right) = (\tan^{-1}2 - \tan^{-1}1)$$

$$+ (\tan^{-1}3 - \tan^{-1}2) + \dots + (\tan^{-1}(n+1) - \tan^{-1}n)$$

$$= \tan^{-1}(n+1) - \tan^{-1}1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \tan \left\{ \sum_{r=1}^n \tan^{-1} \left(\frac{1}{1+r+r^2} \right) \right\}$$

$$\lim_{n \rightarrow \infty} \tan \left(\tan^{-1}(n+1) - \tan^{-1}1 \right)$$

$$= \tan \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = 1$$

19. Answer (3)

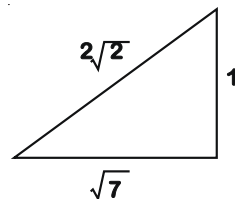
$$\tan \left(\frac{1}{4} \sin^{-1} \frac{\sqrt{63}}{8} \right) \Rightarrow \tan \left(\frac{\theta}{4} \right) = ?$$

Let

$$\sin^{-1} \frac{\sqrt{63}}{8} = \theta \text{ and } \cos \theta = \frac{1}{8} \Rightarrow \sin \theta = \frac{\sqrt{63}}{8} \Rightarrow \tan \theta = \sqrt{63} \text{ and } 6$$

$$\Rightarrow 2 \cos^2 \frac{\theta}{2} - 1 = \frac{1}{8} \Rightarrow 2 \cos^2 \frac{\theta}{2} = \frac{9}{8} \Rightarrow \cos^2 \frac{\theta}{2} = \frac{9}{16}$$

$$\Rightarrow \cos \frac{\theta}{2} = \frac{3}{4}$$



$$\Rightarrow 2 \cos^2 \frac{\theta}{4} - 1 = \frac{3}{4}$$

$$\Rightarrow 2 \cos^2 \frac{\theta}{4} = \frac{7}{4}$$

$$\Rightarrow \cos^2 \frac{\theta}{4} = \frac{7}{8} \Rightarrow \cos \frac{\theta}{4} = \frac{\sqrt{7}}{2\sqrt{2}}$$

$$\Rightarrow \tan \frac{\theta}{4} = \frac{1}{\sqrt{7}}$$

20. Answer (1)

$$\operatorname{cosec} \left(2 \cot^{-1} 5 + \cos^{-1} \left(\frac{4}{5} \right) \right)$$

$$\Rightarrow \operatorname{cosec} \left(\tan^{-1} \left(\frac{5}{12} \right) + \tan^{-1} \left(\frac{3}{4} \right) \right)$$

$$\Rightarrow \operatorname{cosec} \left(\tan^{-1} \left(\frac{56}{33} \right) \right)$$

$$\Rightarrow \operatorname{cosec} \left(\operatorname{cosec}^{-1} \left(\frac{65}{56} \right) \right) = \frac{65}{56}$$

21. Answer (3)

$$\therefore \frac{\sin^{-1} x}{a} = \frac{\cos^{-1} x}{b} = \frac{\tan^{-1} y}{c} = k \text{ (say)}$$

$$\therefore \sin^{-1} x = ak, \cos^{-1} x = bk \text{ and } \tan^{-1} y = ck$$

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$(a+b)x = \frac{\pi}{2}$$

$$\therefore k = \frac{\pi}{2(a+b)}$$

$$\text{Now } \tan^{-1} y = \frac{\pi c}{2(a+b)}$$

$$\therefore \cos \left(\frac{\pi c}{ab} \right) = \cos(2 \tan^{-1} y)$$

$$= \cos \left(\cos^{-1} \left(\frac{1-y^2}{1+y^2} \right) \right) \quad [\text{if } y > 0]$$

$$= \frac{1-y^2}{1+y^2}$$

22. Answer (4)

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1} \left(\frac{a+b}{a-ab} \right) = \frac{\pi}{4}$$

$$\Rightarrow a + b + ab = 1$$

$$\Rightarrow (1+a)(1+b) = 2$$

Given

$$\left(a - \frac{a^2}{2} + \frac{a^3}{3} + \dots \right) + \left(b - \frac{b^2}{2} + \frac{b^3}{3} + \dots \right)$$

$$\ln(1+a) + \ln(1+b)$$

$$\Rightarrow \ln(1+a)(1+b) = \ln 2$$

23. Answer (1)

$$\sin^{-1} \left(\frac{3x}{5} \right) + \sin^{-1} \left(\frac{4x}{5} \right) = \sin^{-1} x$$

$$\sin^{-1} \left(\frac{3x}{5} \sqrt{1 - \frac{16x^2}{25}} + \frac{4x}{5} \sqrt{1 - \frac{9x^2}{25}} \right) = \sin^{-1} x$$

$$\sin^{-1} \left(\frac{3x\sqrt{25-16x^2} + 4x\sqrt{25-9x^2}}{25} \right) = \sin^{-1} x$$

$$3x\sqrt{25-16x^2} + 4x\sqrt{25-9x^2} = 25x$$

$$x = 0 \text{ or } 3\sqrt{25-16x^2} + 4\sqrt{25-9x^2} = 25$$

Squaring both sides

$$x^2 = \frac{1}{2}$$

$$x = \pm \frac{1}{\sqrt{2}}$$

$$\therefore x = 0, \pm \frac{1}{\sqrt{2}}$$

24. Answer (1)

$$\cot^{-1}(\alpha) = \cot^{-1} 2 + \cot^{-1} 8 + \cot^{-1} 18 + \cot^{-1} 32 + \dots 100 \text{ terms}$$

$$= \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{18} + \tan^{-1} \frac{1}{32} + \dots 100 \text{ term}$$

$$= \sum_{k=1}^{100} \tan^{-1} \frac{1}{2k^2}$$

$$= \sum_{k=1}^{100} \tan^{-1} \frac{2}{4k^2} = \sum_{k=1}^n \tan^{-1} \frac{(2k+1) - (2k-1)}{1 + (2k-1)(2k+1)}$$

$$= \sum_{k=1}^{100} \left(\tan^{-1}(2k+1) - \tan^{-1}(2k-1) \right)$$

$$= \tan^{-1} 201 - \tan^{-1} 1$$

$$= \tan^{-1} \frac{200}{202}$$

$$= \cot^{-1}(1.01)$$

$$\text{Hence } \alpha = 1.01$$

25. Answer (3)

$$\tan^{-1}(x+1) + \cot^{-1} \left(\frac{1}{x-1} \right) = \tan^{-1} \left(\frac{8}{31} \right)$$

$$\Rightarrow \tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1} \frac{8}{31}$$

$$\Rightarrow \tan^{-1} \left(\frac{(x+1) + (x-1)}{1 - (x+1)(x-1)} \right) = \tan^{-1} \frac{8}{31}$$

$$\Rightarrow \frac{x}{2-x^2} = \frac{4}{31}$$

$$\Rightarrow 4x^2 + 31x - 8 = 0 \Rightarrow x = \frac{1}{4} \text{ or } x = -8$$

$$x = \frac{1}{4} \text{ does not satisfy}$$

$$\text{Hence, sum of possible values of } x = -8 = \frac{-32}{4}$$

26. Answer (4)

$$\sin^{-1} \left(\left[x^2 + \frac{1}{3} \right] \right) + \cos^{-1} \left(\left[x^2 + \frac{1}{3} \right] - 1 \right) = x^2$$

$$\therefore x^2 + \frac{1}{3} \in \left[\frac{1}{3}, \frac{4}{3} \right]; \text{ so } \left[x^2 + \frac{1}{3} \right] = 0 \text{ or } 1$$

Hence L.H.S. is always equal to π .

and $x^2 = \pi$ has no solution in $[-1, 1]$.

27. Answer (1)

$$\operatorname{cosec}^{-1} x \text{ defined for } x \in (-\infty, -1] \cup [1, \infty)$$

$$\text{also } \sqrt{\{x\}} > 0 \Rightarrow x \neq \mathbb{Z}$$

$$\therefore f(x) \text{ is defined for all non-integer except interval } [-1, 1]$$

28. Answer (4)

$$x(x+1) \geq 0 \quad \dots(i)$$

$$x^2 + x + 1 \leq 1$$

$$\Rightarrow x^2 + x \leq 0 \quad \dots(ii)$$

$$\Rightarrow x^2 + x = 0 \Rightarrow x = 0, -1$$

When $x = 0$ or -1

$$\text{LHS} = \frac{\pi}{2} \Rightarrow \text{No solution}$$

29. Answer (1)

$$2 \tan^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{\frac{6}{5}}{1 - \frac{9}{25}}\right) = \tan^{-1}\left(\frac{\frac{6}{5}}{\frac{16}{25}}\right) = \tan^{-1}\frac{15}{8}$$

$$2 \tan^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \tan^{-1}\left(\frac{15}{8}\right) + \tan^{-1}\left(\frac{5}{12}\right)$$

$$= \tan^{-1}\left(\frac{\frac{15}{8} + \frac{5}{12}}{1 - \frac{15}{8} \cdot \frac{5}{12}}\right)$$

$$= \tan^{-1}\left(\frac{180 + 40}{21}\right) = \tan^{-1}\left(\frac{220}{21}\right)$$

30. Answer (2)

For domain

$$\frac{1+x}{x} \leq -1 \text{ or } \frac{1+x}{x} \geq 1$$

$$\Rightarrow \frac{1+2x}{x} \leq 0 \text{ or } \frac{1}{x} \geq 0$$

$$\Rightarrow x \in \left[-\frac{1}{2}, 0\right) \text{ or } x \in (0, \infty)$$

$$\therefore \text{domain } x \in \left[-\frac{1}{2}, 0\right) \cup (0, \infty)$$

$$\text{i.e. } x \in \left[-\frac{1}{2}, \infty\right) - \{0\}$$

31. Answer (4)

$$(\sin^{-1} x)^2 - (\cos^{-1} x)^2 = (\sin^{-1} x + \cos^{-1} x)(\sin^{-1} x - \cos^{-1} x) = a$$

$$= \frac{\pi}{2} \left(\frac{\pi}{2} - 2\cos^{-1} x \right) = a$$

$$\Rightarrow \frac{\pi}{2} - 2\cos^{-1} x = \frac{2a}{\pi}$$

take sine both sides

$$\therefore \sin\left(\frac{\pi}{2} - 2\cos^{-1} x\right) = \sin\left(\frac{2a}{\pi}\right)$$

$$\Rightarrow \cos(2\cos^{-1} x) = \sin\left(\frac{2a}{\pi}\right)$$

$$\Rightarrow 2\cos^2(\cos^{-1} x) - 1 = \sin\left(\frac{2a}{\pi}\right)$$

$$\Rightarrow 2x^2 - 1 = \sin\left(\frac{2a}{\pi}\right)$$

32. Answer (1)

$$\therefore \sqrt{1 + \sin x} = \sin \frac{x}{2} + \cos \frac{x}{2} \text{ and}$$

$$\sqrt{1 - \sin x} = \sin \frac{x}{2} - \cos \frac{x}{2}$$

$$y(x) = \cot^{-1} \left(\frac{2\sin \frac{x}{2}}{2\cos \frac{x}{2}} \right) = \cot^{-1} \left(\tan \frac{x}{2} \right) = \frac{\pi}{2} - \frac{x}{2}$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

33. Answer (4)

$$\text{Range of } \sin x + \cos x \text{ for } x \in \left[0, \frac{\pi}{2}\right] \text{ is } [1, \sqrt{2}]$$

$$\text{So, } M = \tan^{-1} \sqrt{2} \text{ and } m = \tan^{-1} 1$$

$$\Rightarrow M - m = \tan^{-1} \left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right)$$

$$\Rightarrow \tan(M - m) = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} = 3 - 2\sqrt{2}$$

34. Answer (3)

$$\therefore \left(\frac{3x^2 + x - 1}{(x-1)^2} \right) \in [-1, 1] \text{ and } \frac{x-1}{x+1} \in [-1, 1]$$

$$\Rightarrow x \in \left[-2, \frac{1}{2}\right] \text{ and } x \in (-\infty, 0] \cup \left[\frac{1}{4}, \infty\right) - \{1\} \text{ and}$$

$$x \in [0, \infty)$$

$$\text{finally } x \in \{0\} \cup \left[\frac{1}{4}, \frac{1}{2}\right]$$

35. Answer (3)

$$\cos^{-1}(\cos(-5)) = -5 + 2\pi = a(\text{say})$$

$$\sin^{-1}(\sin 6) = 6 - 2\pi = b(\text{say})$$

$$\tan^{-1}(\tan 12) = 12 - 4\pi = c(\text{say})$$

$$\therefore a + b - c = 2\pi - 5 + 6 - 2\pi - 12 + 4\pi = 4\pi - 11$$

36. Answer (3)

$$\text{Let } T_k = \tan^{-1} \left(\frac{6^r}{2^{2r+1} + 3^{2r+1}} \right) = \tan^{-1} \left(\frac{\frac{2^r}{3^{r+1}}}{1 + \left(\frac{2}{3}\right)^{2r+1}} \right)$$

$$= \tan^{-1} \left(\frac{\left(\frac{2}{3}\right)^r - \left(\frac{2}{3}\right)^{r+1}}{1 + \left(\frac{2}{3}\right)^{2r+1}} \right) = \tan^{-1} \left(\frac{2}{3} \right)^r - \tan^{-1} \left(\frac{2}{3} \right)^{r+1}$$

$$\text{then } S_k = \sum_{r=1}^k T_k = \tan^{-1} \left(\frac{2}{3} \right) - \tan^{-1} \left(\frac{2}{3} \right)^{k+1}$$

$$\lim_{k \rightarrow \infty} S_k = \tan^{-1} \left(\frac{2}{3} \right).$$

37. Answer (2)

$$\therefore \tan^{-1} \frac{1}{2r^2} = \tan^{-1} \left(\frac{2}{1+(4r^2-1)} \right)$$

$$= \tan^{-1} \left(\frac{(2r+1)-(2r-1)}{1+(2r+1)(2r-1)} \right)$$

$$= \tan^{-1}(2r+1) - \tan^{-1}(2r-1)$$

$$\therefore \sum_{r=1}^{50} \tan^{-1} \left(\frac{1}{2r^2} \right) = \sum_{r=1}^{50} (\tan^{-1}(2r+1) - \tan^{-1}(2r-1))$$

$$\therefore p = \tan^{-1}(101) - \tan^{-1}1$$

$$= \tan^{-1} \left(\frac{101-1}{1+101 \cdot 1} \right)$$

$$\therefore \tan p = \frac{50}{51}$$

38. Answer (1)

$$\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left(\frac{5x}{1-6x^2} \right) = \frac{\pi}{4} \Rightarrow \frac{5x}{1-6x^2} = 1$$

$$\text{i.e. } 6x^2 + 5x - 1 = 0$$

$$(6x-1)(x+1) = 0$$

$$\Rightarrow x = \frac{1}{6} \quad (\text{as } x \geq 0)$$

Hence A is a singleton set

39. Answer (1)

$$\text{Let } \tan^{-1}x = t \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$$

$$\cot^{-1}x = \frac{\pi}{2} - t$$

$$f(t) = t^3 + \left(\frac{\pi}{2} - t \right)^3 \Rightarrow f'(t) = 3t^2 - 3 \left(\frac{\pi}{2} - t \right)^2$$

$$f'(t) = 0 \text{ at } t = \frac{\pi}{4}$$

$$f(t)|_{\min} = \frac{\pi^3}{64} + \frac{\pi^3}{64} = \frac{\pi^3}{32}$$

Max will occur around $t = -\frac{\pi}{2}$

$$\text{Range of } f(t) = \left[\frac{\pi^3}{32}, \frac{7\pi^3}{8} \right]$$

$$k \in \left[\frac{1}{32}, \frac{7}{8} \right]$$

40. Answer (4)

$$-1 \leq \frac{x^2 - 5x + 6}{x^2 - 9} \leq 1 \text{ and } x^2 - 3x + 2 > 0, \neq 1$$

$$\frac{(x-3)(2x+1)}{x^2-9} \geq 0 \mid \frac{5(x-3)}{x^2-9} \geq 0$$

Solution to this inequality is

$$x \in \left[-\frac{1}{2}, \infty \right) - \{3\}$$

For $x^2 - 3x + 2 > 0$ and $\neq 1$

$$x \in (-\infty, 1) \cup (2, \infty) - \left\{ \frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2} \right\}$$

Combining the two solution sets (taking intersection)

$$x \in \left[-\frac{1}{2}, 1 \right) \cup (2, \infty) - \left\{ \frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2} \right\}$$

41. Answer (2)

Given $x * y = x^2 + y^3$ and $(x * 1) * 1 = x * (1 * 1)$

$$\text{So, } (x^2 + 1) * 1 = x * 2$$

$$\Rightarrow (x^2 + 1)^2 + 1 = x^2 + 8$$

$$\Rightarrow x^4 + 2x^2 + 2 = x^2 + 8$$

$$\Rightarrow (x^2)^2 + x^2 - 6 = 0$$

$$\therefore (x^2 + 3)(x^2 - 2) = 0$$

$$\therefore \boxed{x^2 = 2}$$

$$\text{Now, } 2 \sin^{-1} \left(\frac{x^4 + x^2 - 2}{x^4 + x^2 + 2} \right) = 2 \sin^{-1} \left(\frac{4}{8} \right)$$

$$= 2 \cdot \frac{\pi}{6} = \frac{\pi}{3}$$

42. Answer (2)

$$\begin{aligned} & \tan^{-1} \left(\frac{\cos\left(\frac{15\pi}{4}\right) - 1}{\sin\frac{\pi}{4}} \right) \\ &= \tan^{-1} \left(\frac{\frac{1}{\sqrt{2}} - 1}{\frac{1}{\sqrt{2}}} \right) \\ &= \tan^{-1}(1 - \sqrt{2}) = -\tan^{-1}(\sqrt{2} - 1) \\ &= -\frac{\pi}{8} \end{aligned}$$

43. Answer (2)

$$f(x) = 2\cos^{-1}x + 4\cot^{-1}x - 3x^2 - 2x + 10 \quad \forall x \in [-1, 1]$$

$$\Rightarrow f'(x) = -\frac{2}{\sqrt{1-x^2}} - \frac{4}{1+x^2} - 6x - 2 < 0 \quad \forall x \in [-1, 1]$$

So, $f(x)$ is decreasing function and range of $f(x)$ is

$$[f(1), f(-1)], \text{ which is } [\pi + 5, 5\pi + 9]$$

$$\begin{aligned} \text{Now } 4a - b &= 4(\pi + 5) - (5\pi + 9) \\ &= 11 - \pi \end{aligned}$$

44. Answer (3)

$$\begin{aligned} & \cos^{-1} \left(\frac{3}{10} \cos \left(\tan^{-1} \left(\frac{4}{3} \right) \right) + \frac{2}{5} \sin \left(\tan^{-1} \left(\frac{4}{3} \right) \right) \right) \\ &= \cos^{-1} \left(\frac{3}{10} \cdot \frac{3}{5} + \frac{2}{5} \cdot \frac{4}{5} \right) \\ &= \cos^{-1} \left(\frac{1}{2} \right) = \frac{\pi}{3} \end{aligned}$$

45. Answer (1)

$$\begin{aligned} & \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) + \cos^{-1} \left(\frac{-\sqrt{3}}{2} \right) + \tan^{-1}(-1) \\ &= \frac{\pi}{3} + \frac{5\pi}{6} - \frac{\pi}{4} \\ &= \frac{4\pi + 10\pi - 3\pi}{12} = \frac{11\pi}{12} \end{aligned}$$

46. Answer (1)

$$\begin{aligned} & \cot \left(\sum_{n=1}^{50} \tan^{-1} \left(\frac{1}{1+n+n^2} \right) \right) \\ &= \cot \left(\sum_{n=1}^{50} \tan^{-1} \left(\frac{(n+1)-n}{1+(n+1)n} \right) \right) \\ &= \cot \left(\sum_{n=1}^{50} (\tan^{-1}(n+1) - \tan^{-1}n) \right) \\ &= \cot(\tan^{-1}51 - \tan^{-1}1) \\ &= \cot \left(\tan^{-1} \left(\frac{51-1}{1+51} \right) \right) \end{aligned}$$

$$\begin{aligned} &= \cot \left(\cot^{-1} \left(\frac{52}{50} \right) \right) \\ &= \frac{26}{25} \end{aligned}$$

47. Answer (3)

$$\begin{aligned} & \lim_{n \rightarrow \infty} 6 \tan \left\{ \sum_{r=1}^n \tan^{-1} \left(\frac{1}{r^2 + 3r + 3} \right) \right\} \\ &= \lim_{n \rightarrow \infty} 6 \tan \left\{ \sum_{r=1}^n \tan^{-1} \left(\frac{(r+2) - (r+1)}{1 + (r+2)(r+1)} \right) \right\} \\ &= \lim_{n \rightarrow \infty} 6 \tan \left\{ \sum_{r=1}^n (\tan^{-1}(r+2) - \tan^{-1}(r+1)) \right\} \\ &= \lim_{n \rightarrow \infty} 6 \tan \left\{ \tan^{-1}(n+2) - \tan^{-1}2 \right\} \\ &= 6 \tan \left\{ \frac{\pi}{2} - \cot^{-1} \left(\frac{1}{2} \right) \right\} \\ &= 6 \tan \left(\tan^{-1} \left(\frac{1}{2} \right) \right) \\ &= 3 \end{aligned}$$

48. Answer (4)

$$-1 \leq \frac{2}{\pi} \sin^{-1} \left(\frac{1}{4x^2 - 1} \right) \leq 1$$

$$-\frac{\pi}{2} \leq \sin^{-1} \frac{1}{4x^2 - 1} \leq \frac{\pi}{2}$$

$$-1 \leq \frac{1}{4x^2 - 1} \leq 1$$

$$\frac{1}{4x^2 - 1} \geq -1 \Rightarrow \frac{4x^2}{(2n+1)(2x-1)} \geq 0$$

$$\begin{array}{ccccccc} & + & & - & & - & + \\ & | & & | & & | & | \\ & -\frac{1}{2} & & 0 & & \frac{1}{2} & \end{array}$$

$$x \in \left(-\infty, -\frac{1}{2} \right) \cup (0) \cup \left(\frac{1}{2}, \infty \right)$$

... (1)

$$\frac{1}{4x^2 - 1} \leq 1 \Rightarrow 1 - \frac{1}{4x^2 - 1} \geq 0 \Rightarrow \frac{4x^2 - 2}{4x^2 - 1} \geq 0$$

$$\Rightarrow \frac{\left(x + \frac{1}{\sqrt{2}} \right) \left(x - \frac{1}{\sqrt{2}} \right)}{\left(x + \frac{1}{2} \right) \left(x - \frac{1}{2} \right)} \geq 0$$

$$\begin{array}{ccccccc} & + & & - & & + & - & & + \\ & | & & | & & | & | & & | \\ & -\frac{1}{\sqrt{2}} & & -\frac{1}{2} & & \frac{1}{2} & \frac{1}{\sqrt{2}} & \end{array}$$

$$x \in \left(-\infty, -\frac{1}{\sqrt{2}} \right] \cup \left(-\frac{1}{2}, \frac{1}{2} \right) \cup \left(\frac{1}{\sqrt{2}}, \infty \right)$$

... (2)

From (1) & (2),

$$x \in \left(-\infty, -\frac{1}{\sqrt{2}} \right] \cup \left[\frac{1}{\sqrt{2}}, \infty \right) \cup \{0\}$$

49. Answer (29)

$$50 \tan \left(\tan^{-1} \frac{1}{2} + 2 \tan^{-1} \left(\frac{1}{2} \right) + 2 \tan^{-1} (2) \right) + 4\sqrt{2} \tan \left(\frac{\tan^{-1} (2\sqrt{2})}{2} \right)$$

$$\Rightarrow 50 \tan \left(\pi + \tan^{-1} \left(\frac{1}{2} \right) \right) + 4\sqrt{2} \tan \left(\frac{1}{2} \tan^{-1} 2\sqrt{2} \right)$$

$$\Rightarrow 50 \left(\frac{1}{2} \right) + 4\sqrt{2} \tan \alpha, \text{ where } 2\alpha = \tan^{-1} 2\sqrt{2}$$

$$\Rightarrow \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = 2\sqrt{2} \quad \dots (i)$$

$$\Rightarrow 2\sqrt{2} \tan^2 \alpha + 2 \tan \alpha - 2\sqrt{2} = 0$$

$$\Rightarrow 2\sqrt{2} \tan^2 \alpha + 4 \tan \alpha - 2 \tan \alpha - 2\sqrt{2} = 0$$

$$\Rightarrow (2\sqrt{2} \tan \alpha - 2)(\tan \alpha - \sqrt{2}) = 0$$

$$\Rightarrow \tan \alpha = \sqrt{2} \text{ or } \frac{1}{\sqrt{2}}$$

$$\Rightarrow \tan \alpha = \frac{1}{\sqrt{2}}$$

($\tan \alpha = \sqrt{2}$ doesn't satisfy (i))

$$\Rightarrow 25 + 4\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 29$$

50. Answer (2)

$$\tan \left(2 \tan^{-1} \frac{1}{5} + \sec^{-1} \frac{\sqrt{5}}{2} + 2 \tan^{-1} \frac{1}{8} \right)$$

$$= \tan \left(2 \tan^{-1} \left(\frac{\frac{1}{5} + \frac{1}{8}}{1 - \frac{1}{5} \cdot \frac{1}{8}} \right) + \sec^{-1} \frac{\sqrt{5}}{2} \right)$$

$$= \tan \left[2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2} \right]$$

$$= \tan \left[\tan^{-1} \frac{\frac{2}{3}}{1 - \frac{1}{9}} + \tan^{-1} \frac{1}{2} \right]$$

$$= \tan \left[\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{2} \right]$$

$$= \tan \left[\tan^{-1} \frac{\frac{3}{4} + \frac{1}{2}}{1 - \frac{3}{8}} \right] = \tan \left[\tan^{-1} \frac{5}{4} \right]$$

$$= \tan \left[\tan^{-1} 2 \right] = 2$$

51. Answer (12)

$$\cos\left(\sin^{-1}\left(x \cot\left(\tan^{-1}\left(\cos\left(\sin^{-1}\right)\right)\right)\right)\right) = k$$

$$\Rightarrow \cos\left(\sin^{-1}\left(x \cot\left(\tan^{-1}\sqrt{1-x^2}\right)\right)\right) = k$$

$$\Rightarrow \cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right) = k$$

$$\Rightarrow \frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}} = k$$

$$\Rightarrow \frac{1-2x^2}{1-x^2} = k^2$$

$$\Rightarrow 1-2x^2 = k^2 - k^2x^2$$

$$\therefore x^2 - \left(\frac{k^2-1}{k^2-2}\right) = 0 \quad \alpha \quad \beta \quad \star$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = 2\left(\frac{k^2-2}{k^2-1}\right) \quad \dots(1)$$

$$\text{and } \frac{\alpha}{\beta} = -1 \quad \dots(2)$$

$$\therefore 2\left(\frac{k^2-2}{k^2-1}\right)(-1) = -5$$

$$\Rightarrow k^2 = \frac{1}{3}$$

$$\text{and } b = \text{S.R} = 2\left(\frac{k^2-2}{k^2-1}\right) - 1 = 4$$

$$\therefore \frac{b}{k^2} = \frac{4}{\frac{1}{3}} = 12$$

52. Answer (2)

$$-1 \leq \frac{x^2-4x+2}{x^2+3} \leq 1$$

$$\Rightarrow -x^2-3 \leq x^2-4x+2 \leq x^2+3$$

$$\Rightarrow 2x^2-4x+5 \geq 0 \quad \& \quad -4x \leq 1$$

$$x \in \mathbb{R} \quad \& \quad x \geq -\frac{1}{4}$$

$$\text{So domain is } \left[-\frac{1}{4}, \infty\right).$$

53. Answer (1)

$$\cos^{-1}x - 2\sin^{-1}x = \cos^{-1}2x$$

$$\text{For Domain : } x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$$\cos^{-1}x - 2\left(\frac{\pi}{2} - \cos^{-1}x\right) = \cos^{-1}(2x)$$

$$\Rightarrow \cos^{-1}x + 2\cos^{-1}x = \pi + \cos^{-1}2x$$

$$\Rightarrow \cos(3\cos^{-1}x) = -\cos(\cos^{-1}2x)$$

$$\Rightarrow 4x^3 = x$$

$$\Rightarrow x = 0, \pm \frac{1}{2}$$

54. Answer (3)

$$f(x) = \tan^{-1}(\sin x - \cos x), \quad [0, \pi]$$

$$\text{Let } g(x) = \sin x - \cos x$$

$$= \sqrt{2} \sin\left(x - \frac{\pi}{4}\right) \text{ and } x - \frac{\pi}{4} \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

$$\therefore g(x) \in [-1, \sqrt{2}]$$

and $\tan^{-1}x$ is an increasing function

$$\therefore f(x) \in [\tan^{-1}(-1), \tan^{-1}\sqrt{2}]$$

$$\in \left[-\frac{\pi}{4}, \tan^{-1}\sqrt{2}\right]$$

$$\therefore \text{Sum of } f_{\max} \text{ and } f_{\min} = \tan^{-1}\sqrt{2} - \frac{\pi}{4}$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) - \frac{\pi}{4}$$

55. Answer (3)

$$f(x) = \sin^{-1} \left(\frac{x^2 - 3x + 2}{x^2 + 2x + 7} \right)$$

$$-1 \leq \frac{x^2 - 3x + 2}{x^2 + 2x + 7} \leq 1$$

$$\frac{x^2 - 3x + 2}{x^2 + 2x + 7} \leq 1$$

$$x^2 - 3x + 2 \leq x^2 + 2x + 7$$

$$5x \geq -5$$

$$x \geq -1 \quad \dots(i)$$

$$\frac{x^2 - 3x + 2}{x^2 + 2x + 7} \geq -1$$

$$x^2 - 3x + 2 \geq -x^2 - 2x - 7$$

$$2x^2 - x + 9 \geq 0$$

$$x \in R \quad \dots(ii)$$

(i) $\cap$ (ii),

Domain $\in [-1, \infty)$

56. Answer (2)

$$\text{Let } \frac{\sin^{-1} x}{\alpha} = \frac{\cos^{-1} x}{\beta} = k \Rightarrow \sin^{-1} x + \cos^{-1} x = k(\alpha + \beta) \quad \text{Without Limits}$$

$$\Rightarrow \alpha + \beta = \frac{\pi}{2k}$$

$$\text{Now, } \frac{2\pi\alpha}{\alpha + \beta} = \frac{2\pi\alpha}{\frac{\pi}{2k}} = 4k\alpha = 4\sin^{-1} x$$

$$\text{Here } \sin\left(\frac{2\pi\alpha}{\alpha + \beta}\right) = \sin(4\sin^{-1} x)$$

$$\text{Let } \sin^{-1} x = \theta \quad \therefore x \in \left(0, \frac{1}{\sqrt{2}}\right) \Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$$

$$\Rightarrow x = \sin \theta$$

$$\Rightarrow \cos \theta = \sqrt{1 - x^2}$$

$$\Rightarrow \sin 2\theta = 2x \cdot \sqrt{1 - x^2}$$

$$\Rightarrow \cos 2\theta = \sqrt{1 - 4x^2(1 - x^2)} = \sqrt{(2x^2 - 1)^2} = 1 - 2x^2$$

□ □ □

$$\left(\because \cos 2\theta > 0 \text{ as } 2\theta \in \left(0, \frac{\pi}{2}\right) \right)$$

$$\Rightarrow \sin 4\theta = 2 \cdot 2x\sqrt{1 - x^2} (1 - 2x^2)$$

$$= 4x\sqrt{1 - x^2} (1 - 2x^2)$$

57. Answer (3)

$$-1 \leq 2x^2 - 3 < 2 \quad \log_1(x^2 - 5x + 5) > 0$$

$$\text{or } 2 \leq 2x^2 < 5 \quad 0 < x^2 - 5x + 5 < 1$$

$$\text{or } 1 \leq x^2 < \frac{5}{2} \quad x^2 - 5x + 5 > 0 \text{ \& } x^2 - 5x + 4 < 0$$

$$x \in \left(-\sqrt{\frac{5}{2}}, -1\right] \quad x \in \left(-\infty, \frac{5 - \sqrt{5}}{2}\right) \cup \left(\frac{5 + \sqrt{5}}{2}, \infty\right)$$

$$\cup \left[1, \sqrt{\frac{5}{2}}\right) \quad \& \quad x \in (-\infty, 1) \cup (4, \infty)$$

Taking intersection

$$x \in \left(1, \frac{5 - \sqrt{5}}{2}\right)$$

58. Answer (130)

$$x = \sin(2\tan^{-1} \alpha) = \frac{2\alpha}{1 + \alpha^2} \quad \dots(i)$$

$$\text{and } y = \sin\left(\frac{1}{2}\tan^{-1} \frac{4}{3}\right) = \sin\left(\sin^{-1} \frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}$$

$$\text{Now, } y^2 = 1 - x$$

$$\frac{1}{5} = 1 - \frac{2\alpha}{1 + \alpha^2}$$

$$\Rightarrow 1 + \alpha^2 = 5 + 5\alpha^2 - 10\alpha$$

$$\Rightarrow 2\alpha^2 - 5\alpha + 2 = 0$$

$$\therefore \alpha = 2, \frac{1}{2}$$

$$\therefore \sum_{\alpha \in S} 16\alpha^3 = 16 \times 2^3 + 16 \times \frac{1}{2^3}$$

$$= 130$$